Neutral Geometry

April 18, 2013

1 Geometry without parallel axiom

Let l, m be two distinct lines cut by a third line t at point P on l and point Q on m. Let A be a point on l and B a point on m such that A, B are on the same side of t. Let A' be a point on the opposite open ray of $\mathring{r}(P, A)$, and B' a point on the opposite open ray of $\mathring{r}(Q, B)$. The four angles $\angle APQ$, $\angle A'PQ$, $\angle BQP$, $\angle B'QP$ are known as **interior angles**. The pair $\angle APQ, \angle B'QP$ and the other pair $\angle A'PQ, \angle BQP$ are known as two pairs of **alternate interior angles**. See Figure 1.

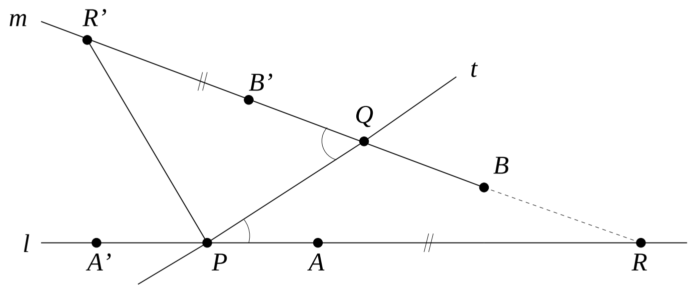

Figure 1: Congruent alternate interior angles imply parallel

Theorem 1.1 (**Alternate Interior Angle Theorem).** *If two distinct lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are parallel.*

Proof. Let l, m be two lines cut by a transversal t such that $\angle APQ \cong \angle B'QP$; see Figure 1. We need to show that l, m do not meet. Suppose they meet at a point R, say, on the side of A, B bounded by t. Draw a point R' on the ray $r(Q, B')$ so that $QR' \cong PR$. Then $\Delta QPR \cong \Delta PQR'$ by SAS. Thus $\angle PQR \cong \angle QPR'$, i.e., $\angle BQP \cong \angle R'PQ$.

Since $\angle A'PQ$ is supplementary to $\angle APQ$, and $\angle BQP$ is supplementary to $\angle B'QP$, we see that $\angle A'PQ \cong \angle BQP$ by Supplementary Angle Congruence Rule. So $\angle A'PQ \cong \angle R'PQ$. This means that R' must be on the line l by Betweenness Axiom 1. Then it forces that $l = m$ since there is only one line through two points R, R'. This is a contradiction. Hence l, m do not meet, i.e., $l \| m$. $\square$

Corollary 1.2. *Tow distinct lines perpendicular to a common line are parallel. In particular, there is exactly one line through a point P perpendicular to a given line l.*

Proof. Let l, m be two distinct lines perpendicular to a common line t. If l, m are cut by t at two distinct points P, Q respectively, then all interior angles are right angles. Hence they are congruent by Right Angle Congruence Theorem. In particular, the alternate interior angles are congruent. So l, m are parallel by Alternate Interior Angle Theorem 1.1. If l, m are cut by t at the same point, we must have $l = m$, since all right angles are congruent and the two lines perpendicular to t must be the same.

In the case that P is not on the line l, suppose there are two lines m, n perpendicular to l and both pass through P. Then m, n are parallel by Alternate Interior Angle Theorem 1.1. This is contradictory to the fact that they meet at P. In the case that P is on l, the perpendicular line to l through P is unique because all right angles are congruent and two perpendicular lines must be the same. $\square$

Corollary 1.3. *Through a point P not on a line l there exists at least one line m parallel to l.*

Proof. Through the point P not on line l there is a unique line m perpendicular to l. Through P there is a unique line n perpendicular to m. The line m cut both lines l, n and all interior angles are right angles, hence they are congruent. Of course, the alternate interior angles are congruent. So l, n are parallel by Alternate Interior Angle Theorem. $\square$

Warning. Uniqueness of perpendicular line does not imply the uniqueness of parallel line.

Recall the **exterior angle** of a triangle and its **remote exterior angles**.

Theorem 1.4 (Exterior Angle Theorem). *An exterior angle of a triangle is greater than each of its remote interior angles.*

Proof. Given a triangle ΔABC and its exterior angle $\angle BCD$ at vertex C. See Figure 2. We show $\angle B < \angle BCD$ first. Suppose $\angle ABC \cong \angle BCD$. Then lines $\overline{AB}, \overline{AC}$ are parallel (since alternate interior angles are congruent). This is impossible.

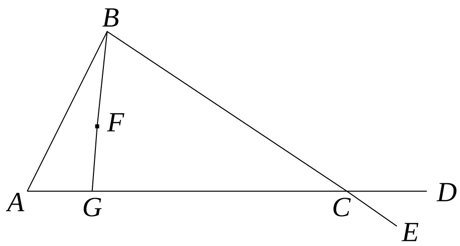

Figure 2: Exterior angle is larger than its remote exterior angle

Suppose $\angle ABC > \angle BCD$. Take an interior point F of $\angle ABC$ such that $\angle FBC \cong \angle BCD$. Then lines $\overline{BF}, \overline{CD}$ are parallel (since alternate interior angles are congruent). However, $\overline{BF}$ meets AC at G between A and C by Between-Cross Lemma. Since AC is part of line $\overline{CD}$, this is contradictory to that $\overline{BF}, \overline{CD}$ are parallel. Hence we must have $\angle ABC < \angle BCD$.

Likewise, the exterior angle $\angle ACE > \angle BAC$. Since $\angle ACE \cong \angle BCD$ as opposite angles, we have $\angle BCD > \angle BAC$ by linear order property of angles. $\square$

Proposition 1.5 (Side-angle-angle criterion) (SAA). *Given triangles ΔABC and $\Delta A'B'C'$. If $AB \cong A'B'$, $\angle BAC \cong \angle B'A'C'$, and $\angle ACB \cong \angle A'C'B'$, then $\Delta ABC \cong \Delta A'B'C'$.*

Proof. Suppose $\angle ABC > \angle A'B'C'$. Pick an interior point D of $\angle ABC$ such that $\angle ABD \cong \angle A'B'C'$. The line $\overline{BD}$ meets AC at E between A and C by Between-Cross Lemma. Then $\Delta ABE \cong \Delta A'B'C'$ by ASA. Thus $\angle AEB \cong \angle A'C'B'$. Since $\angle A'C'B' \cong \angle ACB$, then $\angle AEB \cong \angle ACB$. Note that $\angle AEB$ is an exterior angle of ΔBCE at E. Then $\angle AEB > \angle ECB$, i.e., $\angle AEB > \angle ACB$ by Exterior Angle Theorem. This is a contradiction.

Suppose $\angle ABC < \angle A'B'C'$. A similar argument leads to a contradiction. $\square$

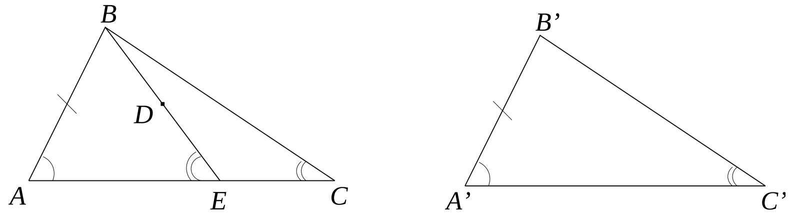

Figure 3: Side-angle-angle criterion

Warning. There is no angle-side-side criterion for congruence of triangles. Here is an example of two non-congruent triangles satisfying the angle-side-side conditions; see Figure 4.

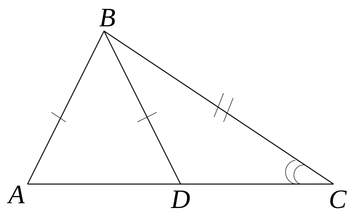

Figure 4: No angle-side-side criterion

Proposition 1.6. *Given right triangles ΔABC and $\Delta A'B'C'$ with right angles $\angle C$ and $\angle C'$. If $AB \cong A'B'$ and $BC \cong B'C'$, then $\Delta ABC \cong \Delta A'B'C'$.*

Proof. Suppose $AC > A'C'$. Let E be a point between A and C such that $EC \cong A'C'$. See Figure 5. Then $\Delta EBC \cong \Delta A'B'C'$ by SAS. Subsequently, $EB \cong A'B' \cong AB$. This means that ΔBAE is an isosceles triangle. Hence $\angle EAB \cong \angle AEB$. Now Exterior Angle Theorem

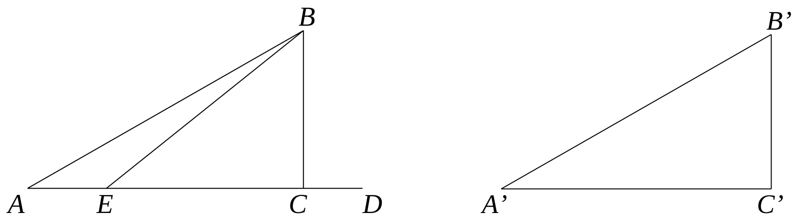

Figure 5: Right triangle congruence criterion

and right angle property imply

$$\angle AEB > \angle ECB \cong \angle DCB > \angle CAB = \angle EAB \cong \angle AEB,$$

i.e., $\angle AEB > \angle AEB$, which is a contradiction. Likewise, $AC < A'C'$ leads to a similar contradiction. We then have $AC \cong A'C'$. Hence $\Delta ABC \cong \Delta A'B'C'$ by SAS. $\square$

A **midpoint** of a segment AB is a point P between A and B such that $AP \cong PB$. It is easy to see that the midpoint of a segment is unique if it exists.

Proposition 1.7 (Midpoint Theorem). *Every segment has a unique midpoint.*

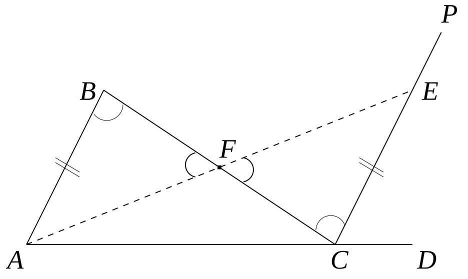

Figure 6: Construction of midpoint

Proof. Given a triangle ΔABC. We plan to find the midpoint of segment BC. Extend AC to a point D such that $A * C * D$. Then A, D are on opposite sides of line $\overline{BC}$. Since $\angle DCB > \angle ABC$ by Exterior Angle Theorem, there exists a unique ray $r(C, P)$ between rays $r(C, B), r(C, D)$ such that $\angle ABC \cong \angle BCP$. Note that P, D are on the same side of $\overline{BC}$. Mark the unique point E on $r(C, P)$ such that $AB \cong CE$ by Congruence Axiom 1. Note that D, E, P are the same side $\overline{BC}$. Since A, D are on opposite sides of $\overline{BC}$, then A, E are on opposite sides of $\overline{BC}$. So line $\overline{BC}$ meets the segment AE at a point F between A and E. To see that F is between B and C, it suffices to show that $E \in \mathring{\angle} BAC$ because of Crossbar Theorem. In fact, E, P, B are on the same side of $\overline{AC}$. Since $\angle ABC \cong \angle BCE$, line $\overline{CE}$ is parallel to $\overline{AB}$; of course, E, P, C are the same side of $\overline{AB}$. Then $E \in \mathring{\angle} BAC$.

Now we have $AB \cong EC$, $\angle ABC \cong \angle ECB$, and $\angle AFB \cong EFC$ (opposite angles), then $\Delta ABF \cong ECF$ by SAA. Hence $BF \cong CF$. The uniqueness is trivial. $\square$

Proposition 1.8 (Bisector Theorem). *(a) Every angle has a unique bisector. (b) Every segment has a unique perpendicular bisector.*

Proof. (a) Given an angle $\angle AOB$ with $OA \cong OB$. Draw the segment AB, find the midpoint P of AB, and draw segment OP. Then $\Delta OPA \cong \Delta OPB$ by SSS. Hence $\angle AOP \cong \angle BOP$. So ray $r(O, P)$ is a bisector of $\angle AOB$. The uniqueness is trivial.

(b) Trivial. $\square$

Proposition 1.9 (Angle-opposite-side relation). *For any triangle ΔABC, we have $\angle A > \angle B$ if and only if $BC > AC$.*

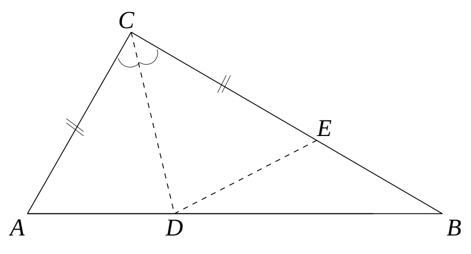

Figure 7: Side-angle proportion relation

Proof. "$\Leftarrow$" (sufficiency): Find bisector $r(C, P)$ of $\angle C$, which meets AB at D between A and B. Lay off CA on CB to have segment $CE \cong CA$. Then $\Delta ACD \cong \Delta ECD$ by SAS. So $\angle A \cong \angle CED$. Note that $\angle CED > \angle B$ by Exterior Angle Theorem for ΔBDE. Hence $\angle A > \angle B$.

"$\Rightarrow$" (necessity): If $BC \cong AC$, then ΔABC is an isosceles triangle. So $\angle A \cong \angle B$, which is contradictory to $\angle A > \angle B$. If $BC < AC$, then $\angle B > \angle A$, which is contradictory to $\angle A > \angle B$. Hence $BC > AC$ by trichotomy of segments. $\square$

Corollary 1.10. *(a) If $\angle Q$ is a right angle for a triangle ΔPQR, then $\angle P < \angle Q$, $\angle R < \angle Q$.*

(b) Given a point P not on a line l, let Q be the foot of the perpendicular line through P. Then $PQ < PR$ for all points R on l other than Q.

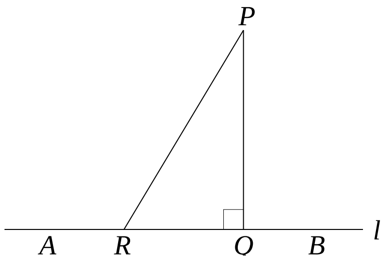

Figure 8: Shortest distance from a point to a line

Proof. Mark two points A, B on line l such that $A * R * Q * B$. Then $\angle BQP > \angle QRP$ by Exterior Angle Theorem for ΔRQP. Since $\angle BQP \cong \angle RQP$, then $\angle QRP < \angle RQP$. Subsequently, $PQ < PR$ by the angle-opposite-side relation for triangle ΔRQP. $\square$

2 Measures of segments and angles

It is clear that segment congruence is an equivalence relation on the set of all segments. For each segment AB we may use $[AB]$ to denote the congruence equivalence class of AB. Recall that line $\overline{AB}$ is totally ordered. We may view AB as a segment together with an order such that $A \prec B$. For any segment CD, we can construct a new segment AP on the ray $r(A, B)$ by laying off CD on the ray $r(B, B')$, where $A * B * B'$, such that $BB' \cong CD$. We write this new segment AB' as

$$AB + CD.$$

Note that $CD + AB$ is a segment on ray $r(C, D)$ with endpoint C. Of course, $AB + CD$ and $CD + AB$ are certainly distinct segments. However, $AB + CD \cong CD + AB$. For the congruence equivalence classes $[AB], [CD]$, we define

$$[AB] + [CD] := [AB + CD].$$

If EF is a segment, it is easy to see that

$$(AB + CD) + EF \cong AB + (CD + EF).$$

We then have

$$[AB] + [CD] = [CD] + [AB].$$

$$([AB] + [CD]) + [EF] = [AB] + ([CD] + [EF]).$$

Let $\frac{1}{2} \cdot AB$ denote the segment whose endpoints are A and the midpoint of AB. For each positive integer k, let $k \cdot AB$ denote the segment obtained by laying off k copies of AB on the ray $r(A, B)$ starting from A; let us define the segment

$$\frac{1}{2^k} \cdot AB := \underbrace{\frac{1}{2} \cdot \left(\cdots \left(\frac{1}{2} \cdot \left(\frac{1}{2} \cdot AB \right) \right) \right)}_{k}.$$

For integers $p \in \mathbb{Z}$ and positive integer $q \in \mathbb{Z}_+$, we write

$$2^p q \cdot AB := 2^p \cdot (q \cdot AB) = q \cdot (2^p \cdot AB),$$

$$2^p q[AB] := [2^p q \cdot AB].$$

Theorem 2.1. *Fix a segment OI, called* **unit segment**. *There exists a unique mapping from the set of all segments to the set $\mathbb{R}_+$ of positive real numbers, $AB \mapsto |AB|$, satisfying the properties:*

(a) $|OI| = 1$.

(b) $|AB| = |CD|$ if and only if $AB \cong CD$.

(c) $A * B * C$ if and only if $|AC| = |AB| + |BC|$.

(d) $|AB| < |CD|$ if and only if $AB < CD$.

(e) For each positive real number a, there exists a segment AB such that $|AB| = a$.

Proof. (Sketch) Fix open ray $\mathring{r}(O, I)$. Each segment is congruent to a segment OA with unique point $A \in \mathring{r}(O, I)$. It suffices to assign for each point $A \in \mathring{r}(O, I)$ a positive real number. The right endpoints of the segments $2^p q \cdot OI$ are assigned to numbers $2^p q$, known as **dyatic rational numbers**. If Archimedes' axiom is satisfied, then every point in $\mathring{r}(O, I)$ has a decimal expression with base 2, and the point is assigned to the real number with decimal expression.

Given a real number $a > 0$. Let $\Sigma_1 := \{P \in \mathring{r}(O, I) : |OQ| \leq a\}$ and $\Sigma_2 := \mathring{r}(O, I) \smallsetminus \Sigma_1$. Then $\{\Sigma_1, \Sigma_2\}$ is a Dedekind cut of $\mathring{r}(O, I)$. There exists a unique Q such that $\Sigma_1 = [O, Q]$ and $\Sigma_2 = (Q, \cdot)$. We must have $|OQ| = a$. $\square$

Definition 1. (a) Two angles $\angle AOB$ and $\angle A'O'B'$ are said to be **addable** if there exists an angle $\angle AOC$ such that $B \in \mathring{\angle} AOC$ and $\angle BOC \cong \angle A'O'B'$. We define the **partial addition**

$$\angle AOB + \angle B'O'C' := \angle AOC.$$

(b) An half-plane is also known as a **flat angle**. We assume that all flat angles are congruent.

Theorem 2.2 (Degree measure of angles)**.** *There exists a unique mapping from the set of all angles to the interval* $(0, 180)$ *of real numbers, $\angle A \mapsto \angle A^\circ$, satisfying the properties:*

(a) $\angle A^\circ = 90^\circ$ if $\angle A$ is a right angle.

(b) $\angle A^\circ = \angle B^\circ$ if and only if $\angle A \cong \angle B$.

(c) If $h(O, B)$ is contained in $\angle^o AOC$, then $\angle AOC^\circ = \angle AOB^\circ + \angle BOC^\circ$.

(d) $\angle A^\circ < \angle B^\circ$ if and only if $\angle A < \angle B$.

(e) If $\angle A, \angle B$ are complementary, then $\angle A^\circ + \angle B^\circ = 180^\circ$.

(f) For each positive real number $a \in$ (0, 180), there exists an angle $\angle AOB$ such that $\angle AOB^\circ = 180^\circ$.

Proof. (Sketch) Fix ray $r(O, A)$ and consider angles $\angle AOB$. Since each angle can be bisected, we denote by $\frac{1}{2} \cdot \angle AOB$ the angle $\angle AOP$, where $r(O, P)$ is the bisector of $\angle AOB$. We then have angles

$$\frac{1}{2^k} \cdot \angle AOB := \underbrace{\frac{1}{2} \cdot \left(\cdots \left(\frac{1}{2} \cdot \left(\frac{1}{2} \cdot \angle AOB \right) \right) \right)}_{k}, \quad k \geq 1.$$

If $\angle AOB$ is addable to $(k-1)\angle AOB$, we define

$$k\angle AOB := (k-1)\angle AOB + \angle AOB$$

We thus have

$$2^{-p} q \, \angle AOB = q(2^{-p} \angle AOB)$$

as long as they are addable to each other. Let $[\angle AOB]$ denote the equivalence class of angles congruent to $\angle AOB$. Then

$$2^{-p} q \, [\angle AOB] := [2^{-p} q \, \angle AOB].$$

Now we fix a right angle $\angle ROT$. Assign the angles $2^{-p}q\,\angle ROT$ to the real numbers $90 \cdot 2^{-p}q$, that is,

$$2^{-p}q\,\angle ROT \mapsto (2^{-p}q\,\angle ROT)^\circ = 90 \cdot 2^{-p}q^\circ.$$

Dedekind's axiom ensures that the assignment can be extended into a bijection from angle congruence classes onto the open interval $(0, 180)$ of real numbers expressed in decimal form of base 2, satisfying (a)-(f). For instance, for part (e) about complementary angles $\angle A, \angle B$, we see that $\frac{1}{2}\angle A + \frac{1}{2}\angle B$ is a right angle so that

$$\left(\frac{1}{2}\angle A + \frac{1}{2}\angle B\right)^\circ = 90^\circ.$$

$\square$

Corollary 2.3 (Consecutive Interior Angle Theorem). *If two distinct lines cut by a transversal have a pair of consecutive interior angles whose angle sum is a flat angle, then the two lines are parallel.*

Proof. We assume in Figure 1 that $\angle APQ + \angle BQP$ is congruent to a flat angle. Since $\angle B'QP + \angle BQP$ is congruent is a flat angle, we see that $\angle APQ \cong \angle B'QP$. So $m \| l$. $\square$

Proposition 2.4 (Triangle inequality). *Let A, B, C be three distinct non-collinear points. Then*

$$|AC| < |AB| + |BC|.$$

Proof. There exists a unique point D such that $A * B * D$ and $BD \cong BC$ by Congruence Axiom 1. See Figure 9. Then $\angle BCD \cong BDC$ because of the isosceles triangle ΔBCD. Since $\angle ACD > \angle BCD \cong \angle BDC$, we have $\angle ACD > \angle BDC = \angle ADC$. Hence $AD > AC$ by the angle-opposite-side relation. Since $AD \cong AB + BC$, we obtain $AB + BC > AC$. Subsequently, $|AB| + |BC| > |AC|$ by measure of segments. $\square$

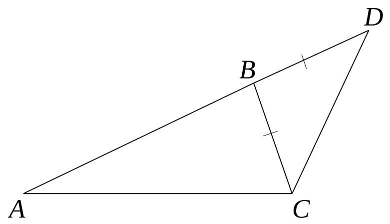

Figure 9: Triangle inequality

Proposition 2.5 (Included angle-opposite-side relation of equal sides). *Given triangles $\Delta ABC, \Delta A'B'C'$ and $AB \cong A'B', AC \cong A'C'$. Then $\angle A > \angle A'$ if and only if $BC > B'C'$.*

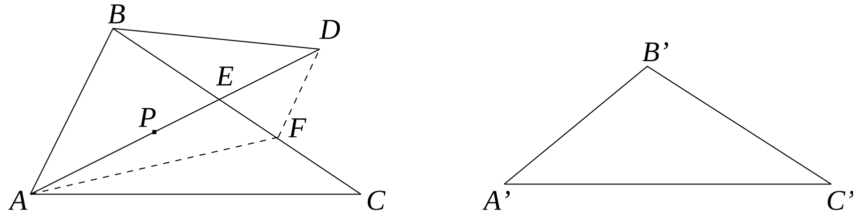

Figure 10: Equal sides and its included angle-opposite-side relation

Proof. "⇒": Let $\angle BAC > \angle B'A'C'$. Draw a ray $r(A,P)$ between $r(A,B)$ and $r(A,C)$ such that $\angle BAP \cong \angle B'A'C'$. Let D be a point on $r(A,P)$ such that $AD \cong A'C'$, and E the intersection of $r(A,P)$ and BC such that $B*E*C$. Draw the angle bisector of $\angle CAD$ and its intersection with CE at F between C and E. Then $\Delta ACF \cong \Delta ADF$ by SAS. Hence $FC \cong FD$. Note that $BF+FD > BD$ by triangle inequality. Then $BF+FC > BD$. Since $BF+FC \cong BC$ and $B'C' \cong BD$. We obtain $BC > B'C'$.

"⇐": Let $BC > B'C'$. If $\angle BAC \cong \angle B'A'C'$; then $\Delta ABC \cong \Delta A'B'C'$ by SAS; so $BC \cong B'C'$, which is a contradiction. If $\angle BAC < \angle B'A'C'$, then $B'C' > BC$ by what just proved previously, which is also a contradiction. Hence we have $\angle BAC > \angle B'A'C'$ by trichotomy of angles. □

3 Saccheri-Legendre theorem

Lemma 2. *Given a triangle ΔABC. Let D be the midpoint of BC and E a point on the ray $r(A,D)$ such that $DE \cong AD$ and $A*D*E$. Draw the segment CE. Then*

(a) The angle sum of ΔAEC equals the angle sum of ΔABC.

(b) Either $\angle EAC \le \frac{1}{2}\angle BAC$ or $\angle AEC \le \frac{1}{2}\angle BAC$.

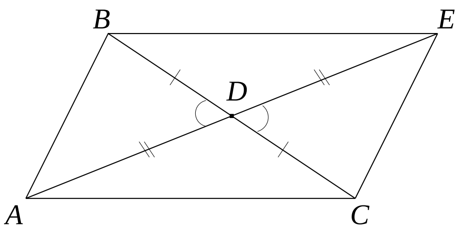

Figure 11: Half-angle triangle

Proof. (a) Note that $\Delta ABD \cong \Delta ECD$ by SAS. Then $\angle DAB \cong \angle DEC$, $\angle DBA \cong \angle DCE$, i.e., $\angle EAB \cong \angle AEC$, $\angle ABC \cong \angle BCE$. The angle sum of ΔAEC is

$$\angle EAC^\circ + \angle AEC^\circ + \angle ACB^\circ + \angle BCE^\circ = \angle EAC^\circ + \angle EAB^\circ + \angle ACB^\circ + \angle ABC^\circ,$$

which is the angle sum of ΔABC.

(b) Since $\angle BAC \cong \angle BAE + \angle CAE$ and $\angle BAE \cong \angle AEC$, we have either $\angle CAE \lesssim \frac{1}{2}\angle BAC$ or $\angle AEC \lesssim \frac{1}{2}\angle BAC$. □

Lemma 3. *The sum of any two angles of a triangle is smaller than a flat angle.*

Proof. Given a triangle ΔABC. The exterior angle of $\angle B$ is larger than $\angle A$. Since the angle sum of $\angle B$ and its exterior angle is a flat angle, then $\angle A$ and $\angle B$ are addable, and their addition is smaller than a flat angle. So their angle sum is less than 180°. □

Theorem 3.1 (Saccheri-Legendre) (Angle-Sum Theorem)**.** *The sum of degree measures of the three angles in any triangle is less than or equal to* 180°.

Proof. Suppose the angle sum of a triangle ΔABC is greater than 180°, say, $180^\circ + \varepsilon^\circ$ with $\varepsilon > 0$. Then by Lemma 2 there exists another triangle $\Delta A_1B_1C_1$, having the angle sum $180^\circ + \varepsilon^\circ$ and $\angle A_1{}^\circ \le \frac{1}{2}\angle A^\circ$. Then there is a triangle $\Delta A_2B_2C_2$, having the angle sum $180^\circ + \varepsilon^\circ$ and $\angle A_2{}^\circ \le \frac{1}{2}\angle A_1{}^\circ$, i.e., $\angle A_2{}^\circ \le \frac{1}{2^2}\angle A^\circ$. Continue this procedure, we obtain triangles $\Delta A_kB_kC_k$ having the angle sum $180^\circ + \varepsilon^\circ$ and $\angle A_k{}^\circ \le \frac{1}{2^k}\angle A^\circ$, $k \ge 1$. When k is large enough we have $\angle A_k{}^\circ < \varepsilon^\circ$, then the sum of the other two angles of $\Delta A_kB_kC_k$ will be greater than 180°, which is contradictory to that the sum of any two angles of a triangle is less than 180°. □

Corollary 3.2. *The sum of two angles of a triangle is less than or equal to its remote exterior angle.*

Proof. Given a triangle ΔABC. Let $\operatorname{ext}(\angle C)$ denote the exterior angle of ΔABC at C. Then

$$\angle A + \angle B + \angle C \le 108^\circ = \angle C + \operatorname{ext}(\angle C).$$

So $\angle A + \angle B \le \operatorname{ext}(\angle C)$. $\square$

Definition 4. A **quadrilateral** is a collection of four points A, B, C, D, denoted $\square ABCD$, such that no three of them are collinear, and the interior of the four segments AB, BC, CD, DA are disjoint. A quadrilateral is said to be **convex** if it has a pair of opposite sides, say, AB and CD, such that CD is contained in an open half-plane bounded by $\overline{AB}$, and AB is contained in an open half-plane bounded by $\overline{CD}$.

Lemma 5. *A quadrilateral $\square ABCD$ is convex if and only if the intersection*

$$\mathring{\square} ABCD := \mathring{\angle} ABC \cap \mathring{\angle} BCD \cap \mathring{\angle} CDA \cap \mathring{\angle} DAB,$$

called the **interior** *of $\square ABCD$, is nonempty. If $\square ABCD$ is convex, then AC meets BD at $P \in \mathring{\square} ABCD$. We define*

$$\square ABCD := \angle ABC \cap \angle BCD \cap \angle CDA \cap \angle DAB.$$

Proof. "$\Leftarrow$:" Take a point $P \in \mathring{\square} ABCD$. Then C, P are the on the same side of $\overline{AB}$; D, P are the on the same side of $\overline{AB}$. So C, D are the same side of $\overline{AB}$. Likewise, A, B are the same side of $\overline{CD}$. Hence $\square ABCD$ is convex by definition.

"$\Rightarrow$:" Let A, B be on the same side of $\overline{CD}$, and C, D be on the same side of $\overline{AB}$. Draw segment AC to have triangles ΔABC and ΔCDA. Then either B, D are on opposite sides of $\overline{AC}$ or B, D are on the same side of $\overline{AC}$.

Case 1. Points B, D are on opposite sides of $\overline{CA}$. Then $\overline{CA}$ meets BD at a point P such that $B * P * D$. There are three subcases.

Subcase 1.1. $C * P * A$. This means that AC intersects BD at P such that $A * P * C$ and $B * P * D$. Since $P \in AC$, we have $P \in \mathring{\angle} ABC$ and $P \in \mathring{\angle} CDA$. Since $P \in BD$, we have $P \in \mathring{\angle} DAB$ and $P \in \mathring{\angle} BCD$. Hence $P \in \mathring{\square} ABCD$. See the left of Figure 12.

Subcase 1.2. $P * C * A$. Then $P \in \mathring{\angle} DAB$ by Crossbar Theorem. Thus $C \in \mathring{\Delta} DAB$. Hence ray $r(D, C)$ intersects AB at Q such that $A * Q * B$. This is contradictory to that A, B are on the same side of $\overline{CD}$. See the middle of Figure 12.

Subcase 1.3. $C * A * P$. Then $P \in \mathring{\angle} BCD$ by Crossbar Theorem. Thus $A \in \mathring{\Delta} BCD$. Hence ray $r(B, A)$ intersects CD at Q such that $C * Q * D$. This is contradictory to that C, D are on the same side of $\overline{AB}$. See the right of Figure 12.

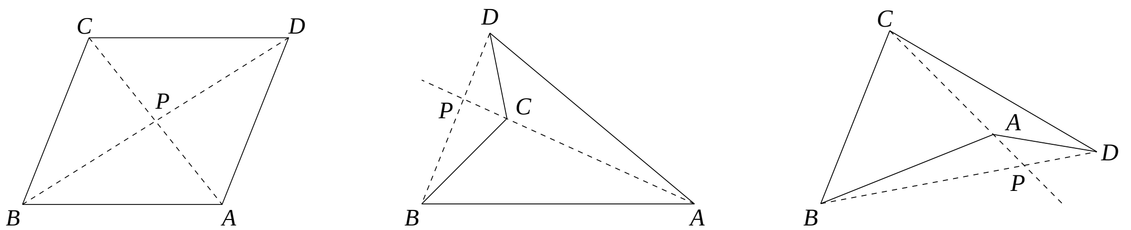

Figure 12: Convex quadrilateral and possible cases impossible

Case 2. Points B, D are on the same side of $\overline{AC}$. Then either $r(A, D)$ is between $r(A, B)$ and $r(A, C)$, or $r(A, B)$ is between $r(A, D)$ and $r(A, C)$.

Subcase 2.1. Ray $r(A, D)$ is between $r(A, B)$ and $r(A, C)$. Then $r(A, D)$ meets BC at a point Q such that $B*Q*C$. If $D*Q*A$, then AD meets BC at Q, which is contradictory to definition of quadrilateral $\square ABCD$. See the left of Figure 13. If $Q*D*A$, then $D \in \mathring{\angle} BAC$. Thus $r(C, D)$ meets AB, which is contradictory to that A, B are on the same side of $\overline{CD}$. See the moddle of Figure 13.

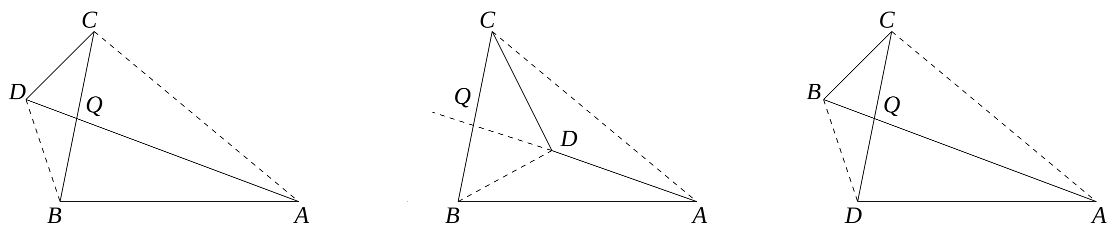

Figure 13: Other possible cases impossible

Subcase 2.2. Ray $r(A, B)$ is between $r(A, D)$ and $r(A, C)$. Then $r(A, B)$ meets CD at a point Q such that $C*Q*D$. Then C, D are on opposite sides of $\overline{AB}$, contradictory to that C, D are on the same side of $\overline{AB}$. See the right of Figure 13. $\square$

Proposition 3.3. *Degree measure of angle sum of convex quadrilateral is at most* $360°$.

Proof. Given a convex quadrilateral $\square ABCD$ and consider triangles ΔABC and ΔCDA by drawing segment AC. We have $\angle BCA + \angle ACD \cong \angle BCD$, $\angle BAC + \angle CAD \cong \angle BAD$. Then the angle sum of $\square ABCD$ is congruent to the addition of the angle sum of ΔABC and the angle sum of ΔCDA. Hence the angle sum of $\square ABCD$ is less than or equal to $360°$. $\square$

4 Angle Defect of Triangle

Definition 6. The **angle defect** (or just **defect**) of a triangle ΔABC is

$$\delta ABC := 180° - \angle A° - \angle B° - \angle C°.$$

Proposition 4.1 (Additivity of angle defect). *Given a triangle ΔABC and point D between A and B on segment AB. Draw segment CD to have triangles ΔACD and ΔBCD. Then*

$$\delta ABC = \delta ACD + \delta BCD.$$

Consequently, the angle sum of ΔABC is equal to $180°$ *if and only if the angle sums of both triangles ΔACD and ΔBCD are equal to* $180°$.

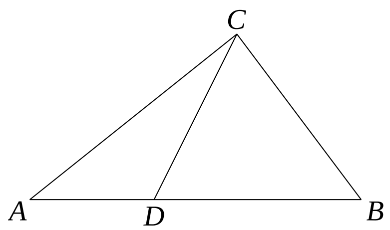

Figure 14: Additivity of angle defect

Proof. Trivial. $\square$

Theorem 4.2 (Once-Then-All Theorem). *If there is one triangle whose angle sum is* $180°$*, then all triangles have angle sum equal to* $180°$*. This can be split into the following three statements.*

(a) If there exists a triangle whose angle sum is $180°$*, then there exists a rectangle.*

(b) If there exists a rectangle, then every right triangle has angle sum equal to $180°$*.*

(c) If all right triangles have angle sum equal to $180°$*, then every triangle has angle sum equal to* $180°$*.*

Proof. Let ΔABC be a triangle. Then ΔABC has at least two acute angles, say, $\angle A$ and $\angle B$. There exists a unique line $\overline{CD}$ perpendicular to $\overline{AB}$, meeting $\overline{AB}$ at D. If $D * A * B$, then $\angle BAC > \angle ADC =$ right angle, which is a contradiction. Likewise, $A * B * D$ is impossible. Then we must have $A * D * B$. Thus ΔACD and ΔBCD are right triangles.

(a) Let ΔABC have angle sum equal to $180°$, i.e., its angle defect is zero. Then both right triangles ΔACD and ΔBCD have angle sum equal $180°$. We construct a rectangle $\Box BDCE$ from the right triangle ΔBCD.

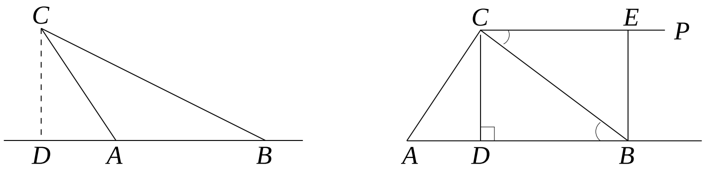

Figure 15: A triangle is cut to two right triangles

Draw ray $r(C, P)$ to be such that $\angle BCP \cong \angle CBD$. Mark a point on $r(C, P)$ such that $CE \cong DB$. Then $\Delta CBD \cong \Delta BCE$ by SAS. Thus $\angle BCD \cong \angle CBE$, $\angle BEC \cong \angle CDB$, so $\angle BEC$ is a right angle. Since $\delta CBD = 0$, so is δBCE. Moreover, $\angle DCB + \angle BCE \cong \angle DCE$ is a right angle. Hence the quadrilateral $\Box BDCE$ is a rectangle.

(b) Given an arbitrary right triangle $\Delta A'B'C'$. It is easy to see that a rectangle can be doubled in either side. So we may assume that there exists a rectangle $\Box ABCD$ such that $AB > A'B'$ and $CB > C'B'$. See Figure 16. Clearly, ΔABC and ΔADC are right triangles, having zero angle defect. Mark points P on AB and Q on BC such that $A * P * B$, $C * Q * B$, and $PB \cong A'B'$, $QB \cong C'B'$. Then $\Delta PBQ \cong \Delta A'B'C'$. Now the triangle ΔPBC has zero angle defect by additivity. Then ΔPBQ has zero angle defect by additivity again. So $\Delta A'B'C'$ has zero angle defect.

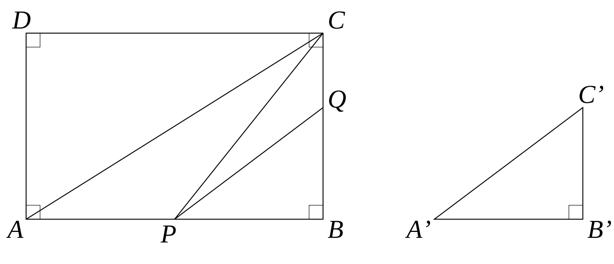

Figure 16: Big rectangle

(c) Now for an arbitrary triangle ΔABC, we may assume that $\angle A$ and $\angle B$ are acute. Then ΔABC can be divided into two right triangles ΔACD and ΔBCD. Then ΔACD and ΔBCD have zero angle defect. So is ΔABC by additivity of angle defect. Hence ΔABC has angle sum of degree measures equal to $180°$. $\Box$

Corollary 4.3. *If a triangle has positive angle defect, then all triangles have positive angle defect.*

Proof. Trivial. $\Box$

5 Equivalence of Parallel Postulates

Unique Parallel Line Postulate (UPLP). Through a point P not on a line l there exists exactly one line m parallel to l.

Euclid's Postulate V (EPV). If two distinct lines l, m intersect a transversal t in such a way that the sum of the interior angles on one side of t is less than a flat angle, then l, m meet on the same side of t having the two interior angles.

Hilbert's Axiom of Parallelism (HAP). Through a point P not on a line l there exists at most one line m parallel to l.

Theorem 5.1. *Euclid's Postulate V $\Leftrightarrow$ Hilbert's Axiom of Parallelism.*

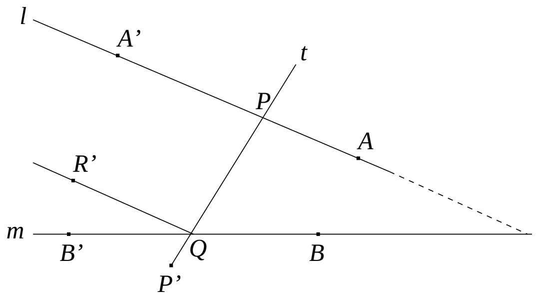

Figure 17: Equivalence of Euclid's Postulate V and Hilbert's axiom

Proof. Given distinct lines l, m, t such that l, t meet at P and m, t meet at Q. Mark point A on l and point B on m on the same side of t. Mark point A' on l and B' on m such that $A' * P * A$ and $B' * Q * B$. Draw ray $r(Q, R')$ on open half-pane $\mathring{H}(t, B')$ such that $\angle R'QP \cong \angle APQ$. Extend $r(Q, R')$ to line m'. Then m' is parallel to l.

"$\Rightarrow$:" Let m be an arbitrary line through Q and parallel to l. EPV implies

$$\angle APQ^\circ + \angle BQP^\circ \geq 180^\circ, \quad \angle A'PQ^\circ + \angle B'QP^\circ \geq 180^\circ.$$

Since $\angle APQ^\circ + \angle BQP + A'PQ^\circ + \angle B'QP = 360^\circ$, it follows that

$$\angle APQ^\circ + \angle BQP^\circ = 180^\circ, \quad \angle A'PQ^\circ + \angle B'QP^\circ = 180^\circ.$$

Since $\angle B'QP^\circ + \angle BQP^\circ = 180^\circ$, then $\angle B'QP^\circ = \angle APQ^\circ$. Note that $\angle APQ = \angle R'QP$. We thus have $\angle BQP = \angle R'QP$. So $m = m'$, which is Hilbert axiom.

"$\Leftarrow$:" Let angle sum of consecutive interior angles be less that 180°, i.e.,

$$\angle APQ^\circ + \angle BQP^\circ < 180^\circ.$$

Since $\angle B'QP^\circ + \angle BQP^\circ = 180^\circ$, then $\angle B'QP^\circ > \angle APQ^\circ$. Clearly, $m' \neq m$. Since $l \subset \mathring{H}(m', P), r(Q, B') \subset \mathring{H}(m', P')$, then $r(P, A'), r(Q, B')$ do not meet. If $r(P, A), r(Q, B)$ do not meet, then m is parallel to l. So $m = m'$, contradiction to $m \neq m'$. Hence $r(P, A), r(Q, B)$ meet. $\square$

Proposition 5.2. *Under Hilbert's Axiom of Parallelism. Let two lines l, m be parallel and cut by a transversal t. Then*

(a) Alternate interior angles are congruent.
(b) Corresponding angles are congruent.
(c) The sum of degree measures of consecutive interior angles is 180°.
(d) The angle sum of degree measures of a triangle is 180°.

Proof. (a) There exists a line m through P and parallel to l, having congruent alternate interior angles. Then every line parallel to l must be this line m. So the alternate interior angles are congruent.

(b) It follows from (a) and the fact that opposite angles are congruent.

(c) It follows from the fact that the sum of supplementary angles is 180°.

(d) Extend the segment AC to E and draw ray $r(C, D)$ such that $\angle BCD \cong \angle B$. Then $r(C, D)$ is parallel to AB. Thus $\angle DCE \cong \angle A$. Hence $\angle A^\circ + \angle B^\circ + \angle C^\circ = \angle DCE^\circ + \angle BCD^\circ + \angle ACB^\circ = 180^\circ$. □

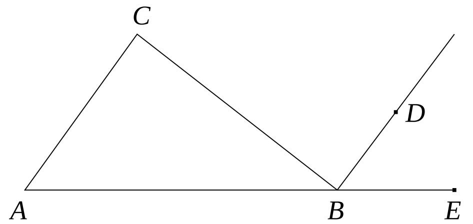

Figure 18: Angle sum of a triangle is a flat angle

Theorem 5.3. *Angle sum of a triangle equal to* $180^\circ \Rightarrow$ *Hilbert's axiom of parallelism.*

Proof. Given a line l and point P not on l. Let t be the unique line through P and perpendicular to l, meeting l at Q. Through P there is a unique line m perpendicular to t. Then m is parallel to l. Let n be an arbitrary line through P and parallel to l, but distinct from m. Pick a point R on m such that n intersects the interior of $\angle QPR$. Fix a point S on n such that $S \in \mathring{\angle} QPY$. Then angle $\angle RPS$ is acute. See Figure 19.

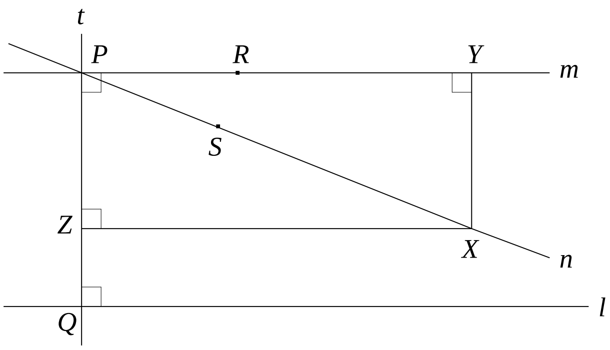

Figure 19: Existence of rectangle implies unique parallel.

Let X be a point on the open ray $\mathring{r}(P, S)$. Drop perpendicular XY to m with foot Y on m, and perpendicular XZ to t with foot Z on t. Note that $l, m, \overline{XZ}$ are parallel to each other. So X, Z are on the same side of lines l, m respectively. We then have $P * Z * Q$. Analogously, lines $t, \overline{XY}$ are parallel to each other. Then X, Y are on the same side of t and P, Z are on the same side of $\overline{XY}$. Hence the quadrilateral $\square PYXZ$ is convex.

Note that $\angle XZP$, $\angle ZPY$, $\angle PYX$ are right angles. Since every triangle has angle sum equal to 180°, then $\angle ZPX + \angle PXZ$ is a right angle. Since $\angle ZPX + \angle XPY$ is a right angle, we see that $\angle XPY \cong \angle PXZ$. Thus $\Delta PXY \cong \Delta XPZ$ by SAA. Subsequently, $PZ \cong XY$.

Now we can take X to be such that $XY > PQ$ by Aristotle's axiom. Then $PZ > PQ$, which is impossible. (Aristotle's axiom can be easily followed from Archimedes' axiom when rectangle exists. However, it is no need to assume the existence of rectangle to obtain Aristotle's axiom from Archimedes' axiom.) □

6 Saccheri quadrilaterals and Lambert quadrilaterals

A quadrilateral is said to be a **Saccheri quadrilateral** if its two base angles are right angles and the base-angle adjacent opposite sides are congruent. For instance, for quadrilateral $\square ABCD$ with right angles $\angle A, \angle B$ and $AD \cong BC$ is a Saccheri quadrilateral.

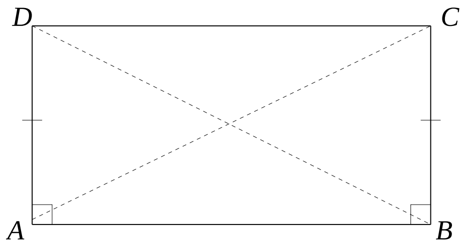

Figure 20: A Saccheri quadrilateral

A quadrilateral is said to be a **Lambert quadrilateral** if it has at least three angles to be right angles.

Proposition 6.1. *Let $\square ABCD$ be a Saccheri quadrilateral with right angles $\angle A, \angle B$, and $AD \cong BC$. See Figure 20. Then $\angle C \cong \angle D$.*

Proof. Draw segments AC and BD. Note that $\Delta ABC \cong \Delta BAD$ by SAS. Then $AC \cong BD$ and $\angle BAC \cong \angle ABD$. Subsequently, $\angle CAD \cong DBC$ by angle subtraction. Hence $\Delta CAD \cong \Delta DBC$ by SAS. Therefore $\angle ADC \cong \angle BCD$. $\square$

Proposition 6.2 (Property of quadrilateral with two adjacent right angles)**.** *Let $\square ABCD$ be a quadrilateral with two adjacent right angles $\angle A$ and $\angle B$. See Figure 21. Then*

(a) $\angle C < \angle D$ if and only if $AD < BC$.

(b) $\angle C \cong \angle D$ if and only if $AD \cong BC$.

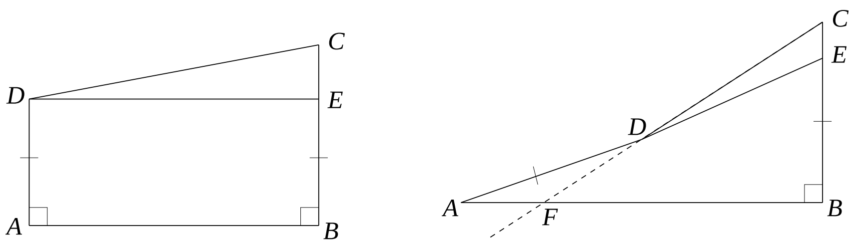

Figure 21: $\square ABED$ is a Saccheri quadrilateral

Proof. (a) "$\Leftarrow$": Assume $AD < BC$. Find a point E on BC such that $BE \cong AD$ and $B * E * C$; see the left of Figure 21. It is clear that B, E, C are on the same side of $\overline{AD}$ for $\overline{BC}$ is parallel to $\overline{AD}$. We claim that A, B are on the same side of $\overline{DC}$. Suppose it is not true, i.e., $\overline{DC}$ meets AB at point F such that $A * F * B$. Note that C, D, F are collinear. Then $\angle AFD$ ($=\angle AFC$) is larger than $\angle B$ and $\angle BFD > \angle A$ by Exterior Angle Theorem. Since $\angle A, \angle B$ are right angles, so the sum of $\angle AFD, \angle BFD$ are not a flat angle, which is a contradiction.

Now we see that B is in the interior of $\angle ADC$. Since $B * E * C$, it follows that E is also in the interior of $\angle ADC$. So the ray $r(D, E)$ is between rays $r(D, A)$ and $r(D, C)$. Hence $\angle ADE \cong \angle BED$. Clearly, Exterior Angle Theorem implies

$$\angle D > \angle ADE \cong \angle BED > \angle C.$$

"⇒": Assume $\angle C < \angle D$. If $AD \cong BC$, then $\angle C \cong \angle D$ by Proposition 6.1, which is contradictory to $\angle C < \angle D$. If $BC < AD$, then $\angle C > \angle D$ by the previous argument, which is a contradictory to $\angle C < \angle D$. So we must have $AD < BC$.

(b) Trivial. □

Proposition 6.3 (Property of quadrilateral with three right angles)**.** *Let $\square ABCD$ be a Lambert quadrilateral with right angles $\angle A, \angle B, \angle C$.*

(a) Then $\angle D$ is never obtuse.

(b) If $\angle D$ is a right angle, then the opposite sides of $\square ABCD$ are congruent.

(c) If $\angle D$ is acute, then each side adjacent to $\angle D$ is greater than its opposite side.

Proof. Assume that AB, CD and AD, BC are two pairs of opposite sides of $\square ABCD$.

(a) Trivial because Lambert quadrilateral is convex and its angle sum is less than or equal to $360°$.

(b) The quadrilateral $\square ABCD$ is a rectangle. Then $AD \cong BC$ and $AB \cong CD$ by the property of quadrilateral with two adjacent right angles.

(c) Then $\angle D < \angle A$ and $\angle D < \angle C$. Hence $AB < DC$ and $CB < DA$ by the property of quadrilateral with two adjacent right angles. □

Lemma 7. *Given a right triangle ΔOXY such that $\angle Y$ is a right angle and $\angle O$ is acute. Extend OX to X' such that $O * X * X'$ and $OX \cong XX'$, extend YX to Z such that $Y * X * Z$ and $\overline{X'Z}$ is perpendicular to $\overline{XY}$, and extend OY to Y' such that $O * Y * Y'$ and $\overline{X'Y'}$ is perpendicular to $\overline{OY}$. See Figure 22. Then*

(a) $X'Y'$ is at least a double of XY, i.e., $X'Y' \geq 2 \cdot XY$.

(b) OY' is at most a double of OY, i.e., $OY' \leq 2 \cdot OY$.

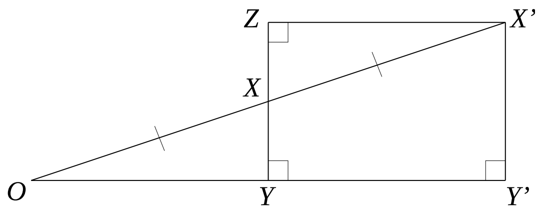

Figure 22: Double of hypotenuse

Proof. Since $XO \cong XX'$, $\angle XYO \cong \angle XZX' =$ right angle, and $\angle OXY \cong \angle X'XZ$, then $\Delta OYX \cong \Delta X'XZ$ by SAA. So $XY \cong XZ$ and $X'Z \cong OY$. Note that $\square YY'X'Z$ is a Lambert quadrilateral with right angles $\angle Y'YZ, \angle YY'X', \angle YZX'$. We have $|X'Y'| \geq |ZY| = 2|XY|$ and $|YY'| \leq |ZX'| = |OY|$ by the property of quadrilateral with three right angles. Hence $|X'Y'| \geq 2|XY|$ and $|OY'| \leq 2|OY|$. □

Archimedes' axiom implies Aristotle's axiom. Given an acute angle $\angle XOY$. For an arbitrary segment AB, there exists a point Y' on ray $r(O, Y)$ such that $X'Y' > AB$, where $X'Y'$ is perpendicular to $\overline{OY}$ with foot Y' on ray $r(O, Y)$.

Proof. Note that there exists a positive integer n such that $2^n \cdot XY > AB$ by Archimedes's axiom. Let X_1 be a point on $r(O, X)$ such that $OX_1 \cong 2 \cdot OX$, and let X_1Y_1 be a segment perpendicular to $\overline{OY}$ with foot Y_1 on $\overline{OY}$. Then $|X_1Y_1| \geq 2|XY|$. Analogously, let X_2 be a point on $r(O, X)$ such that $OX_2 \cong 2 \cdot OX_1$, and let X_2Y_2 be the segment perpendicular to $\overline{OY}$ with foot of X_2 on $\overline{OY}$. Then $|X_2Y_2| \geq 2|X_1Y_1| \geq 2^2 \cdot |XY|$. Continue this procedure,

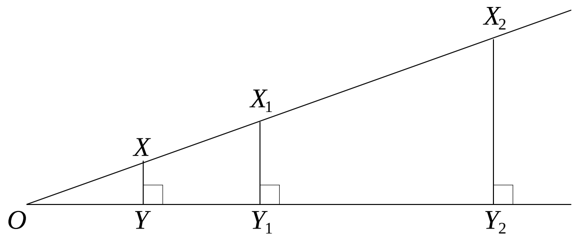

Figure 23: Archimedes' axiom implies Aristotle's axiom

we have points X_k $(k \geq 1)$ on $r(O, X)$ such that $OX_k \cong 2 \cdot OX_{k-1}$, and segments X_kY_k perpendicular to $\overline{OY}$ with foot Y_k on $\overline{OY}$. Then

$$|X_kY_k| \geq 2 \cdot |X_{k-1}Y_{k-1}| \geq 2^k \cdot |XY|, \quad k \geq 1.$$

Hence there exits an integer $n = 2^k$ such that $X_kY_k > n \cdot AB$. $\square$